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THE RECONSTRUCTION PROPERTY IN BANACH SPACES AND A PERTURBATION THEOREM

PETER G. CASAZZA AND OLE CHRISTENSEN

Abstract. Perturbation theory is a fundamental tool in Banach space theory. However, the applications of the classical results are limited by the fact that they force the perturbed sequence to be equivalent to the given sequence. We will develop a more general perturbation theory that does not force equivalence of the sequences.

1. Introduction

Perturbation theory is a very important tool in several areas of mathematics. It began with the fundamental perturbation result by Paley and Wiener [6], stating that a sequence that is sufficiently close to an orthonormal basis in a Hilbert space automatically forms a basis; that is, the *reconstruction property* is preserved. Since then, a number of variations and generalizations of this perturbation theorem have appeared, e.g., to the setting of Banach spaces (see Singer [7], pages 84-109). All of these generalizations have in common that a perturbation $\{g_i\}_{i\in I}$ of a sequence $\{f_i\}_{i\in I}$ in a Banach space X must be equivalent to $\{f_i\}_{i\in I}$; that is, there exists a bounded and invertible operator T on X such that $Tf_i = g_i$ for all $i \in I$. This puts severe restrictions on applications of the theory. In this paper we will present a more general perturbaton theory for reconstruction families in Banach spaces: it is strong enough to capture existing results, but does not force the involved sequences to be equivalent.

2. The Reconstruction Property

We first give a formal definition of the *reconstruction property.*

Definition 2.1. *Let X be a separable Banach space. We say that a sequence $\{f_i^*\}_{i\in I} \subset X^*$ has the reconstruction property for X with respect to a sequence $\{f_i\}_{i\in I} \subset X$ if*

$$f = \sum_{i\in I} f_i^*(f) f_i, \quad \textit{for all } f \in X. \tag{2.1}$$

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In short, we will also say that the pair $\{f_i, f_i^*\}_{i\in I}$ has the reconstruction property for X.

It is important for our applications that $\{f_i\}_{i\in I}$ and $\{f_i^*\}_{i\in I}$ come from X and X^* in Definition 2.1. For example, if $f_i^* \in \ell_\infty$ and $\{f_i^*\}_{i\in I}$ is unitarily equivalent to the unit vector basis of ℓ_2, then this sequence clearly has a "reconstruction" property with respect to its own predual (i.e. expansions with respect to the orthonormal basis) but this family cannot have the reconstruction property with respect to ℓ_1 which is the pre-dual of ℓ_∞. We refer the reader to [2] for a generalization of the reconstruction property.

Remark 2.2. *If $\{f_i, f_i^*\}_{i=1}^\infty$ has the reconstruction property for a Banach space X, then X has the bounded approximation property* [1]*, page 286. In fact, the sequence of finite rank operators $T_n : X \to X,\ T_n f = \sum_{i=1}^n f_i^*(f) f_i$ has the property that $T_n f \to f$ in norm for all $f \in X$. Therefore, X is isomorphic to a complemented subspace of a Banach space with a basis, cf.* [1]*, page 290. Conversely, if X has BAP then there exists a Banach space $X \subset Y$ with a basis $\{f_i, f_i^*\}$ and a projection P of Y onto X. Now, $\{Pf_i, Pf_i^*\}$ has the reconstruction property for X.*

For information on the bounded approximation property, see [1] (Pages 271-316).

We observe that the reconstruction property (2.1) is stronger than the assumption that $\{f_i\}_{i\in I}$ spans the space X:

Proposition 2.3. *There exists a Banach space X with the following properties:*

(i) *There is a sequence $\{f_i\}_{i=1}^\infty$ such that each $f \in X$ has an expansion $f = \sum_{i=1}^\infty a_i f_i$;*
(ii) *X does not have the reconstruction property with respect to any pair $\{h_i, h_i^*\}_{i\in I}$.*

Proof: Let X be a separable Banach space failing the bounded approximation property (see [1] , Chapter 7). Then X does not have the reconstruction property with respect to any family $\{h_i, h_i^*\}_{i\in I}$. Let $T : \ell_1 \to X$ be a quotient map. If $\{e_i\}_{i=1}^\infty$ is the unit vector basis of ℓ_1 let $f_i = Te_i$. If $f \in X$ then there is a $g \in \ell_1$ so that $Tg = f$. Since $g = \sum_{i=1}^\infty g(i)e_i$, we have that

$$f = Tg = \sum_{i=1}^\infty g(i)Te_i = \sum_{i=1}^\infty g(i)f_i.$$

□

Given that $f_i \in X$ satisfies (i) in Proposition 2.3, it would be interesting to find further conditions which guarantee the existence of $f_i^* \in X^*$ so that $\{f_i, f_i^*\}$ has the reconstruction property for X. This however is a very deep question and we do not know the answer even for Hilbert spaces.

3. A Perturbation Theorem

For our main perturbation result we will need several standard results from Banach space theory. We state them in the following lemma. For notation, if X is a Banach space we write B_X for the unit ball of X.

Lemma 3.1. *Let X, Y be Banach spaces and $T : X \to Y$ be a bounded linear operator.*

(1) If T is an isomorphism onto Y, then $\|Tf\| \geq A\|f\|$ for all $f \in X$ if and only if $AB_Y \subset T(B_X)$.

(2) If T is an isomorphism onto Y which satisfies estimates of the form

$$A\|f\| \leq \|Tf\| \leq C\|f\|$$

for all $f \in X$, then for all $g \in Y^*$ we have

$$A\|g\| \leq \|T^*g\| \leq C\|g\|.$$

(3) If T is bounded, linear, and surjective, and $AB_Y \subset T(B_X)$ then T^* is an isomorphism (but not necessarily surjective), satisfying for all $g \in Y^*$ that

$$A\|g\| \leq \|T^*g\| \leq ||T||\, \|g\|.$$

The result below is a Banach space version of the Paley-Wiener theorem for frames in Hilbert space [3].

Theorem 3.2. *Suppose that $\{f_i, f_i^*\}_{i=1}^\infty$ has the reconstruction property for X. Let X_d be a sequence space which has the unit vectors $\{e_i\}_{i=1}^\infty$ as a basis. Assume that*

$$T\{c_i\}_{i=1}^\infty := \sum_{i=1}^\infty c_i f_i$$

defines a bounded linear operator from X_d into X. Assume further that the operator $R : X \to X_d$ given by

$$Rf = \{f_i^*(f)\}_{i=1}^\infty$$

is a bounded operator. Let $\{g_i\}$ be a sequence in X for which there exist constants $\lambda, \mu > 0$ such that $\lambda + \mu\|R\| < 1$ and

$$\left\| \sum_{i=1}^\infty c_i(f_i - g_i) \right\| \leq \lambda \left\| \sum_{i=1}^\infty c_i f_i \right\| + \mu \|\{c_i\}_{i=1}^\infty\|_{X_d}, \tag{3.1}$$

for all finitely non-zero scalar sequences $\{c_i\}_{i=1}^\infty$. Then there are functionals $\{g_i^*\}_{i=1}^\infty \subset X^*$ so that $\{g_i, g_i^*\}_{i=1}^\infty$ has the reconstruction property for X.

Moreover, $U : X_d \to X$ given by $U\{c_i\}_{i=1}^\infty = \sum_{i=1}^\infty c_i g_i$ is a bounded, linear, and surjective operator, and

$$\frac{1}{\|R\|}\left(1 - (\lambda + \mu\|R\|)\right)\|f\| \leq \|U^*f\| \leq \|T\| \left(1 + \lambda + \frac{\mu}{\|T\|}\right)\|f\| \tag{3.2}$$

for all $f \in X^*$. Finally, if the unit vectors form an unconditional basis for X_d, then the series $\sum_{i=1}^{\infty} c_i g_i$ converges unconditionally for all $\{c_i\}_{i=1}^{\infty} \in X_d$.

Proof: For all finite sequences $\{c_i\}_{i=1}^{n}$ we have

$$\|\sum_{i=1}^{n} c_i g_i\| \leq \|\sum_{i=1}^{n} c_i(f_i - g_i)\| + \|\sum_{i=1}^{n} c_i f_i\| \leq (1+\lambda)\|\sum_{i=1}^{n} c_i f_i\| + \mu\|\{c_i\}_{i=1}^{n}\|_{X_d}.$$

It follows that for all $\{c_i\}_{i=1}^{\infty} \in X_d$ and all $n > m$ in $\mathbb{N}$,

(3.3)

$$\|\sum_{i=1}^{n} c_i g_i - \sum_{i=1}^{m} c_i g_i\| = \|\sum_{i=m+1}^{n} c_i g_i\| \leq (1+\lambda)\|\sum_{i=m+1}^{n} c_i f_i\| + \mu\|\sum_{i=m+1}^{n} c_i e_i\|_{X_d}.$$

Since $\sum_{i=1}^{\infty} c_i e_i$ converges by the fact that $\{c_i\}_{i=1}^{\infty} \in X_d$ and the unit vectors form a basis for X_d, it follows that $\sum_{i=1}^{\infty} c_i f_i$ converges by our assumption that T is a bounded operator. Now it follows from (3.3) that $\sum_{i=1}^{\infty} c_i g_i$ converges in X (unconditionally if the unit vectors form an unconditional basis for X_d). If we define: $U : X_d \to X$ by $U\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i g_i$ we have

$$\begin{aligned} \|U\{c_i\}_{i=1}^{\infty}\| &\leq (1+\lambda)\|T\{c_i\}_{i=1}^{\infty}\| + \mu\|\{c_i\}_{i=1}^{\infty}\|_{X_d} \\ &\leq ((1+\lambda)\|T\| + \mu)\, \|\{c_i\}_{i=1}^{\infty}\|_{X_d}. \end{aligned}$$

Hence,

$$\|U\| = \|U^*\| \leq ((1+\lambda)\|T\| + \mu),$$

which verifies the right hand side of (3.2). Next, define an operator $L : X \to X$ by:

$$Lf = \sum_{i=1}^{\infty} f_i^*(f) g_i.$$

For any $f \in X$ we have:

$$\begin{aligned} \|(I-L)f\| &= \|f - Lf\| = \|\sum_{i=1}^{\infty} f_i^*(f) f_i - \sum_{i=1}^{\infty} f_i^*(f) g_i\| \\ &= \|\sum_{i=1}^{\infty} f_i^*(f)(f_i - g_i)\| \\ &\leq \lambda\|\sum_{i=1}^{\infty} f_i^*(f) f_i\| + \mu\|\{f_i^*(f)\}_{i=1}^{\infty}\|_{X_d} \\ &= \lambda\|f\| + \mu\|Rf\|_{X_d} \\ &\leq \lambda\|f\| + \mu\|R\|\|f\| = (\lambda + \mu\|R\|)\|f\|. \end{aligned}$$

Since $\lambda + \mu\|R\| < 1$, it follows that L is an invertible operator on X. Now, let $g_i^* = (L^{-1})^* f_i^*$, for all $i \in \mathbb{N}$. If $f \in X$ we have

$$\sum_{i=1}^{\infty} g_i^*(f) g_i = \sum_{i=1}^{\infty} [(L^{-1})^* f_i^*](f) g_i = \sum_{i=1}^{\infty} f_i^*(L^{-1} f) g_i = L L^{-1} f = f.$$

So $\{g_i, g_i^*\}_{i=1}^{\infty}$ has the reconstruction property for X.

In order to prove the left hand side of (3.2), we note that for $f \in X$

$$\begin{aligned} \|Lf\| &\geq \|f\| - \|(I-L)(f)\| \\ &\geq \|f\| - (\lambda + \mu\|R\|)\|f\| \\ &= (1 - (\lambda + \mu\|R\|))\,\|f\|. \end{aligned}$$

Now consider $U : R(X) \to X$ as defined above. If $f \in B_X$ then

$$\|\{\frac{1}{\|R\|} f_i^*(f)\}_{i=1}^{\infty}\|_{X_d} = \frac{1}{\|R\|}\|R(f)\|_{X_d} \leq \frac{1}{\|R\|}\|R\|\|f\| = 1.$$

So, $\{\frac{1}{\|R\|} f_i^*(f)\}_{i=1}^{\infty} \in B_{X_d}$. Also,

$$\|U\{\frac{1}{\|R\|} f_i^*(f)\}_{i=1}^{\infty}\| = \frac{1}{\|R\|}\|Lf\| \geq \frac{1}{\|R\|}\,(1 - (\lambda + \mu\|R\|))\,\|f\|.$$

By Lemma 3.1 (1),

$$U(B_{X_d}) \supset U(B_{R(X)}) \supset \frac{1}{\|R\|}\,(1 - (\lambda + \mu\|R\|))\,B_X.$$

By Lemma 3.1 (3), we have

$$\|U^* f\| \geq \frac{1}{\|R\|}\,(1 - (\lambda + \mu\|R\|))\,\|f\|.$$

This completes the proof of the theorem. □

We now consider some applications of Theorem 3.2.

Example 3.3. Choose $\lambda, \mu > 0$ so that $\lambda + \mu\|R\| < 1$, (T, R as in Theorem 3.2). Choose any bounded linear operator $L : X_d \to X$ with $\|L\| \leq \|R\|$. For $i \in \mathbb{N}$ let

$$g_i = (1 - \lambda) f_i + \mu L e_i,$$

where $\{e_i\}_{i=1}^\infty$ is a basis of X_d. Then for all $\{c_i\}_{i=1}^\infty \in X_d$ we have

$$\begin{aligned} \|\sum_{i=1}^\infty c_i(f_i - g_i)\| &= \|\lambda \sum_{i=1}^\infty c_i f_i + \mu \sum_{i=1}^\infty c_i L e_i\| \\ &\leq \lambda \|\sum_{i=1}^\infty c_i f_i\| + \mu \|\sum_{i=1}^\infty c_i L e_i\| \\ &\leq \lambda \|\sum_{i=1}^\infty c_i f_i\| + \mu \|L\| \|\{c_i\}_{i=1}^\infty\|_{X_d} \\ &\leq \lambda \|\sum_{i=1}^\infty c_i f_i\| + \mu \|R\| \|\{c_i\}_{i=1}^\infty\|_{X_d} \end{aligned}$$

So the hypotheses of Theorem 3.2 are satisfied.

Another natural application of Theorem 3.2 is to take a Banach space X_d with a basis $\{g_i, g_i^*\}$ and let P be a projection on X_d. Letting $X = P(X_d)$, $f_i = P(g_i)$, $f_i^* = P(g_i^*)$, $T = P$, and R be the injection of X into X_d, we can apply the theorem.

An important aspect of Theorem 3.2 is that it does not require the perturbed family $\{g_i\}$ to be equivalent to the original reconstruction sequence $\{f_i\}$. We will give an example of this below. Recall that two sequences $\{f_i\}_{i\in I}, \{g_i\}_{i\in I}$ in a Banach space are *equivalent* if the mapping $Tf_i := g_i$ can be extended to a well defined bounded linear map of $\overline{\text{span}}\{f_i\}$ onto $\overline{\text{span}}\{g_i\}$.

Proposition 3.4. *There is a Banach space X and a pair $\{f_i, f_i^*\}_{i=1}^\infty$ having the reconstruction property for X, a sequence space X_d with an unconditional basis $\{e_i\}_{i=1}^\infty$ so that the operators T, R in Theorem 3.2 exist and there is a sequence $\{g_i\}_{i=1}^\infty$ in X satisfying the perturbation criterion* (3.1), *but $\{g_i\}_{i=1}^\infty$ is not equivalent to $\{f_i\}_{i=1}^\infty$.*

Proof: Let P be a non-trivial projection on ℓ_p onto a subspace, for any $1 \leq p < \infty$. Let $X_d = \ell_p$, $X = P(X_d)$. With the notation in Theorem 3.2, let $T = P$ and R be the injection of X into X_d. Since X is isomorphic to ℓ_p (See [5]) there is an isomorphism $L : X_d \to X$. Now, choose $\lambda, \mu > 0$ so that

$$\lambda + \mu \; max\{\|L\|, \|R\|\} < 1.$$

Let $\{e_i\}_{i=1}^\infty$ be the unit vector basis of ℓ_p, let $f_i = Pe_i$ for all $i = 1, 2, \cdots$ and let $g_i = (1-\lambda)f_i + \mu L e_i$. For all finitely non-zero sequences $\{c_i\}_{i=1}^\infty$ we have

$$\begin{aligned} \|\sum_{i=1}^\infty c_i(f_i - g_i)\| &\leq \lambda \|\sum_{i=1}^\infty c_i f_i\| + \mu \|\sum_{i=1}^\infty c_i L e_i\| \\ &\leq \lambda \|\sum_{i=1}^\infty c_i f_i\| + \mu \|L\| \|\{c_i\}_{i=1}^\infty\|_{X_d}. \end{aligned}$$

So $\{g_i\}_{i=1}^{\infty}$ is a perturbation of $\{f_i\}_{i=1}^{\infty}$. If we choose any $\{0\} \neq \{c_i\}_{i=1}^{\infty} \in X_d$ so that $\sum_{i=1}^{\infty} c_i P e_i = \sum_{i=1}^{\infty} c_i f_i = 0$ then

$$\sum_{i=1}^{\infty} c_i g_i = (1-\lambda)\sum c_i f_i + \mu \sum_{i=1}^{\infty} c_i L e_i = \mu \sum_{i=1}^{\infty} c_i L e_i.$$

Since $\{Le_i\}_{i=1}^{\infty}$ is a basis for X, it follows that

$$\sum_{i=1}^{\infty} c_i L e_i \neq 0,$$

and so $\{f_i\}_{i=1}^{\infty}$ is not equivalent to $\{g_i\}_{i=1}^{\infty}$. □

We will now show that the conclusion in Theorem 3.2 can be obtained under weaker assumptions. Let us discuss why this is important. In Theorem 3.2, it is easily checked that the operator RT is a projection of X_d onto $R(X)$. This is a pretty strong restriction on the application of the result. As we saw earlier, the very existence of a reconstruction family implies that X is isomorphic to a complemented subspace of a Banach space with a basis. However, the space with a basis may not be the space X_d above. The next result has the advantage that it does not require that X be isomorphic to a complemented subspace of X_d, but just that it embed into X_d. The proof follows line by line the proof of Theorem 3.2 using R^{-1} in place of T.

Theorem 3.5. *Suppose $\{f_i, f_i^*\}_{i=1}^{\infty}$has the reconstruction property for a Banach space X. Let X_d be a sequence space which has the unit vectors as a basis. Assume the operator $R: X \to X_d$ given by*

$$Rf = \{f_i^*(f)\}_{i=1}^{\infty}$$

is a (not necessarily surjective) isomorphism. Let $\{g_i\}_{i=1}^{\infty}$ be a sequence in X for which there exist constants $\lambda, \mu > 0$ such that $\lambda + \mu\|R\| < 1$ and

$$\|\sum_{i=1}^{\infty} c_i(f_i - g_i)\| \leq \lambda \|\sum_{i=1}^{\infty} c_i f_i\| + \mu \|\{c_i\}_{i=1}^{\infty}\|_{X_d},$$

for all finitely non-zero scalar sequences $\{c_i\}_{i=1}^{\infty}$ taken from $\{f_i^*(f)\}_{i=1}^{\infty}$ for any $f \in X$. Then there are functionals $\{g_i^*\}_{i=1}^{\infty} \subset X^*$ so that $\{g_i, g_i^*\}_{i=1}^{\infty}$ has the reconstruction property for X.

Moreover, $U: R(X) \to X$ given by $U\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i g_i$ is an isomorphism satisfying for all $f \in X^*$,

$$\frac{1}{\|R\|}\left(1 - (\lambda + \mu\|R\|)\right)\|f\| \leq \|U^* f\| \leq \|T\|\left((1+\lambda)\|R^{-1}\| + \mu\right)\|f\|.$$

4. The Reconstruction Property Revisited

We will now consider some theoretical consequences of the reconstruction property and related examples.

Proposition 4.1. *Suppose that $\{f_i, f_i^*\}_{i=1}^\infty$ has the reconstruction property for X. Then for all $g \in X^*$ we have that the sequence*

$$\left\{ \sum_{i=1}^n g(f_i) f_i^* \right\}_{n=1}^\infty$$

converges to $g \in X^*$ in the ω^*-topology.

Proof: For any $f \in X$ we have:

$$\begin{aligned} \lim_{n\to\infty} \left[\left(\sum_{i=1}^n g(f_i) f_i^* \right) (f) \right] &= \lim_{n\to\infty} \sum_{i=1}^n g(f_i) f_i^*(f) \\ &= \lim_{n\to\infty} g(\sum_{i=1}^n f_i^*(f) f_i) \\ &= g\left(\lim_{n\to\infty} \sum_{i=1}^n f_i^*(f) f_i \right) = g(f). \end{aligned}$$

This proves the proposition. □

In the case that X is reflexive, the convergence in Proposition 4.1 becomes weak convergence. It is natural to ask whether we also become convergence in norm in this case. Unfortunately, this fails. Even in a Hilbert space, having the reconstruction property with respect to $\{f_i, f_i^*\}$ does not imply the reconstruction property for $\{f_i^*, f_i\}$:

Example 4.2. Let H be a separable Hilbert space. There are vectors $f_i^*, f_i \in H$ so that for every $f \in H$ we have $f = \sum_{i=1}^\infty f_i^*(f) f_i$ but we do not have that $f = \sum_{i=1}^\infty f(f_i) f_i^*$ for all $f \in H$.

To see this, let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis for H, and define the vectors $\{f_i\}_{i=1}^\infty$ and $\{f_i^*\}_{i=1}^\infty$ by

$$f_{2i} = e_i, f_{2i-1} = e_1, \quad f_1^* = e_1, \quad f_{2i}^* = e_i, \quad f_{2i+1}^* = e_{i+1} - e_i.$$

Now, for all $f \in H$,

$$\sum_{i=1}^\infty f_{2i}^*(f) f_{2i} = f$$

and

$$\sum_{i=1}^n f_{2i+1}^*(f) f_{2i+1} = \langle e_1, f \rangle e_1 + \left(\sum_{i=1}^n \langle e_{i+1} - e_i, f \rangle \right) e_1 = \langle e_{n+1}, f \rangle e_1.$$

Since $\lim_{n\to\infty}\langle e_{n+1}, f\rangle = 0$, we have that

$$\sum_{i=1}^{\infty} f_{2i+1}^*(f) f_{2i+1} = 0.$$

Hence, for all $f \in H$,

$$f = \sum_{i=1}^{\infty} f_i^*(f) f_i.$$

On the other hand, if $f = e_1$ then,

$$\begin{aligned}\sum_{i=1}^{2n+1} f(f_i) f_i^* &= \langle e_1, f_2\rangle f_2^* + \sum_{i=0}^{n} \langle e_1, f_{2i+1}\rangle f_{2i+1}^* \\ &= e_1 + \sum_{i=0}^{n} \langle e_1, e_1\rangle (e_{i+1} - e_i) = e_1 + e_{n+1}.\end{aligned}$$

It follows that $\sum_{i=1}^{\infty} f(f_i) f_i^*$ does not converge in H.

The next proposition shows that we can get the reconstruction property with respect to X^* if the reconstruction property for X holds with unconditional convergence.

Proposition 4.3. *Assume that $\{f_i, f_i^*\}_{i=1}^{\infty}$ has the reconstruction property for X and that the series $\sum_{i=1}^{\infty} f_i^*(f) f_i \ \ (= f)$ converges unconditionally for all $f \in X$. Then the following are equivalent:*
(1) For all $g \in X^*$ we have

$$g = \sum_{i=1}^{\infty} g(f_i) f_i^*.$$

(2) c_0 does not embed into X^*.

Proof:
(1) $\Rightarrow$ (2): By (1), X^* is separable and so c_0 cannot embed into X^* [5].
(2) $\Rightarrow$ (1): For $E \subset \mathbb{N}$ finite, define

$$T_E f = \sum_{i\in E} g_i(f) f_i.$$

The family $\{T_E\}$ is a family of finite rank bounded linear operators on X which are pointwise bounded bacause of the unconditional convergence of $\sum_{i=1}^{\infty} f_i^*(f) f_i$. By the Uniform Boundedness Principle, this family is uniformly bounded, i.e.,

$$\sup_{E\subset\mathbb{N}} \|T_E\| = K < \infty.$$

If E, F are finite subsets of $\mathbb{N}$ then

$$\begin{aligned}
\|\sum_{i\in E} g(f_i)f_i^* - \sum_{i\in F} g(f_i)f_i^*\| &= \sup_{\|f\|=1} |\sum_{i\in E} g(f_i)f_i^*(f) - \sum_{i\in F} g(f_i)f_i^*(f)| \\
&\leq \sup_{\|f\|=1} \left(|g(\sum_{i\in E} f_i^*(f)f_i| + \sum_{i\in F} |g(\sum_{i\in F} f_i^*(f)f_i| \right) \\
&\leq \|g\| \left(\sup_{\|f\|=1} \|\sum_{i\in E} f_i^*(f)f_i\| + \sup_{\|f\|=1} \|\sum_{i\in F} f_i^*(f)f_i\| \right) \\
&\leq 2K\,\|g\|\,\|f\|.
\end{aligned}$$

By [4] (Theorem 6 on page 44), it follows that

$$\sum_{i=1}^{\infty} g(f_i)f_i^*$$

is weakly unconditionally Cauchy. Since c_0 does not embed into X^*, by [4] (Theorem 8, page 45), we have that

$$\sum_{i=1}^{\infty} g(f_i)f_i^*$$

is unconditionally convergent in X^*. Since this series converges weakly to g by Proposition 4.1, we have that

$$g = \sum_{i=1}^{\infty} g(f_i)f_i^*$$

and the series converges unconditionally. □

Recall that we say a subspace $Y \subset X^*$ *norms* X if there is a constant $A > 0$ so that for all $f \in X$ we have

$$A\|f\| \leq \sup_{\|g\|=1, g\in Y} |g(f)|.$$

Proposition 4.4. *If* $\{f_i, f_i^*\}_{i=1}^{\infty}$ *has the reconstruction property for* X *, then* $\overline{span}\ \{f_i^*\}_{i=1}^{\infty}$ *norms* X.

Proof: For all $f \in X$ we have

$$f = \sum_i f_i^*(f)f_i.$$

It follows that the finite rank operators

$$T_n(f) = \sum_{i=1}^{n} f_i^*(f)f_i,$$

are pointwise bounded on X. By the Uniform Boundedness Principle, there exists a constant K so that for all n and all $f \in X$,

$$\|\sum_{i=1}^{n} f_i^*(f)f_i\| \le K\|f\|.$$

Now, for every $g \in X^*$ we have,

$$\begin{aligned}
\|\sum_{i=1}^{n} g(f_i)f_i^*\| &= \sup_{\|f\|=1} |(\sum_{i=1}^{n} g(f_i)f_i^*)(f)| \\
&= \sup_{\|f\|=1} |\sum_{i=1}^{n} g(f_i)f_i^*(f)| \\
&= \sup_{\|f\|=1} |g(\sum_{i=1}^{n} f_i^*(f)f_i)| \\
&\le \|g\| \sup_{\|f\|=1} \|\sum_{i=1}^{n} f_i^*(f)f_i\| \\
&\le K\|g\|\ \|f\|.
\end{aligned}$$

Now, fix $f \in X$ $\epsilon > 0$ and choose $g \in X^*$ so that $\|g\| = 1$ and $\|f\| \le (1+\epsilon)|g(f)|$. We have (for n sufficiently large):

$$\begin{aligned}
\|f\| &\le (1+\epsilon)|g(f)| \le (1+\epsilon)|\sum_{i=1}^{\infty} f_i^*(f)g(f_i)| \\
&\le (1+\epsilon)^2|\sum_{i=1}^{n} f_i^*(f)g(f_i)| \le (1+\epsilon)^2|(\sum_{i=1}^{n} g(f_i)f_i^*)(f)|.
\end{aligned}$$

Since $\sum_{i=1}^{n} g(f_i)f_i^*$ is in the closed linear span of the $\{f_i^*\}_{i=1}^{\infty}$ and since the norms of these vectors are uniformly bounded, it follows that the space $\overline{span}\ \{f_i^*\}_{i=1}^{\infty}$ norms X. □

Proposition 4.5. *If $\{f_i, f_i^*\}_{i=1}^{\infty}$ has the reconstruction property for X and $\{f_i^*\}_{i=1}^{\infty}$ is equivalent to $\{g_i^*\}_{i=1}^{\infty}$ in X^*, then $\{g_i^*\}_{i=1}^{\infty}$ has the reconstruction property with respect to some Banach space Y and elements $\{g_i\}_{i=1}^{\infty}$ in Y.*

Proof: Given the isomorphism $Tg_i^* = f_i^*$, let $g_i = T^*f_i$. Note that f_i is a linear functional on the closed linear span of the $\{f_i^*\}$ in an obvious way; but, its norm as a linear functional may not be the same as its norm as an element of X. However, by Proposition 4.4, these norms are equivalent.

Now, for $f \in X$

$$\begin{aligned}\sum_{i=1}^{\infty} g_i^*(f) g_i &= T^* \sum_{i=1}^{\infty} (T^{-1} f_i^*)(f) f_i \\ &= T^* \sum_{i=1}^{\infty} f_i^*((T^{-1})^* f) f_i = T^*(T^{-1})^* f = f.\end{aligned}$$

So $\{g_i, g_i^*\}_{i=1}^{\infty}$ has the reconstruction property for X.

$\square$

We can strengthen the results in the particular case of a reflexive Banach space:

Theorem 4.6. *Let X be reflexive. Assume $\{f_i, f_i^*\}_{i=1}^{\infty}$ has the reconstruciton property for X. Let $\{g_i^*\}_{i=1}^{\infty}$ be elements of X^*. Assume there is a $0 < \lambda < 1$ so that for all $n \in \mathbb{N}$ and all sequences of scalars $\{a_i\}_{i=1}^{n}$ we have*

$$\| \sum_{i=1}^{n} a_i (f_i^* - g_i^*) \| \leq \lambda \| \sum_{i=1}^{n} a_i f_i^* \|.$$

Then there are vectors $\{g_i\}_{i=1}^{\infty} \subset X$ so that for all $f \in X$

$$f = \sum_{i=1}^{\infty} g_i^*(f) g_i.$$

Proof: We first observe that $\overline{span}\ \{f_i^*\}_{i=1}^{\infty} = X^*$. In fact, if this was not the case, the reflexitivity of X is reflexive would imply the existence of an element $f \in X^{**} = X$ so that $f(g) = 0$ for all $g \in \overline{span}\ \{f_i^*\}_{i=1}^{\infty}$; but then

$$\sum_{i=1}^{\infty} f_i^*(f) f_i = 0,$$

which contradicts the assumption that $\{f_i, f_i^*\}_{i=1}^{\infty}$ has the reconstruction property for X. Now, define $T : X^* \to X^*$ by $T(f_i^*) = g_i^*$. Since $\{g_i^*\}_{i=1}^{\infty}$ is a perturbation of $\{f_i^*\}_{i=1}^{\infty}$, this is a well defined operator on X^*. But the perturbation condition implies that $\|I - T\| \leq \lambda < 1$. Hence, T is an invertible operator on X^*, and

$$\sum_{i=1}^{\infty} g_i^*(f)(T^*)^{-1} f_i = (T^*)^{-1} \sum_{i=1}^{\infty} f_i^*(T^* f) f_i = (T^*)^{-1} T^* f = f.$$

So $\{(T^*)^{-1} f_i, g_i^*\}_{i=1}^{\infty}$ has the reconstruction property for X. $\square$

Unfortunately, besides the reflexive case, a perturbation of a family with the reconstruction property need not have the reconstruction property:

Example 4.7. Let $X = c_0$ so $X^* = \ell_1$. Let $\{e_i\}$ (respectively, $\{e_i^*\}$) be the unit vector basis of X (respectively, X^*). Define,

$$f_i^* = e_i^*, \quad f_i = e_i, \quad \text{for all } i = 1, 2, 3, \cdots.$$

Also, let

$$g_1^* = e_1^*, \quad g_i^* = \frac{1}{2}e_1^* + e_i^*, \quad \text{for all } i \geq 2.$$

Of course, $\{f_i^*\}_{i=1}^\infty$ has the reconstruction property with respect to $\{f_i\}_{i=1}^\infty$. Also, for all $n \in \mathbb{N}$ and all families of scalars $\{a_i\}_{i=1}^n$ we have:

$$\|\sum_{i=1}^n a_i(f_i^* - g_i^*)\| = \|\sum_{i=1}^n a_i e_i^* - (\sum_{i=1}^n a_i e_i^* + \frac{1}{2}\sum_{i=2}^n a_i e_1^*)\|$$

$$= \frac{1}{2}\left|\sum_{i=2}^n a_i\right| \leq \frac{1}{2}\sum_{i=1}^n |a_i| = \frac{1}{2}\|\sum_{i=1}^n a_i f_i^*\|.$$

So $\{g_i^*\}_{i=1}^\infty$ is a perturbation of $\{f_i^*\}_{i=1}^\infty$ and hence is a basic sequence in ℓ_1 which is equivalent to the unit vector basis of ℓ_1. But also, $e_i^* = g_i^* - \frac{1}{2}g_1^*$ for all $i = 2, 3, \cdots$. It follows that $\{g_i^*\}_{i=1}^\infty$ is actually a basis for ℓ_1 equivalent to the unit vector basis.

We proceed, by way of contradiction, to show that this family $\{g_i^*\}$ does not have the reconstruction property with respect to any sequence of vectors in c_0. So, assume $\{g_i\}_{i=1}^\infty \subset c_0$ satisfies that

$$f = \sum_{i=1}^\infty g_i^*(f)g_i, \quad \text{for all } f \in c_0.$$

Then, for all $j \geq 2$ we have

$$e_j = \sum_{i=1}^\infty g_i^*(e_j)g_i = g_j.$$

Also,

$$e_1 = \sum_{i=1}^\infty g_i^*(e_i)g_i = g_1 + \sum_{i=2}^\infty \frac{1}{2}g_i = g_1 + \sum_{i=2}^\infty \frac{1}{2}e_i.$$

Hence,

$$g_1 = e_1 - \frac{1}{2}\sum_{i=2}^\infty e_i.$$

It follows that $g_1 \notin c_0$, contradicting our assumption. So $\{g_i^*\}_{i=1}^\infty$ does not have the reconstruction property with respect to any sequence of vectors in c_0.

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